KTH Teknikvetenskap

SF2729 GROUPS AND RINGS
HOMEWORK ASSIGNMENT I
GROUPS

The following homework problems can count as the first problem in the first section of the mid term exam and in the final exam. The solutions should be handed in no later than on February 15, 2011. The computations and arguments should be easy to follow. Collaborations should be clearly stated.

The way the credits from will be counted on the exam according to the following table:

Credits on homework	31-35	26-30	21-25	16-20	11-15	6-10	0-5
Credits on problem 1 of part I	6	5	4	3	2	1	0

The first problem will be divided in three parts and with at least two points from the homework, Part a) cannot give any more points, with at least four points from the homework, Part a) and Part b) cannot give any more points.

PROBLEMS

Problem 1. Show that $[A, B] = AB - BA$ defines a binary operation on the set of real skew-symmetric $n \times n$-matrices, for any positive integer n.

Furthermore, show that for $n = 3$, this binary structure is isomorphic to the binary structure given by the vector product on $\mathbb{R}^3$ by exhibiting an explicit isomorphism. **(5)**

Problem 2. Verify that the two sets of matrices $\{A_0, A_1, \ldots, A_{n-1}, B_0, B_1, \ldots, B_{n-1}\} \subseteq \mathrm{GL}_2(\mathbb{R})$ and $\{C_0, C_1, \ldots, C_{n-1}, D_0, D_1, \ldots, D_{n-1}\} \subseteq \mathrm{GL}_2(\mathbb{C})$ form isomorphic groups, where

$$A_j = \begin{pmatrix} \cos j\phi & -\sin j\phi \\ \sin j\phi & \cos j\phi \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} \sin j\phi & \cos j\phi \\ \cos j\phi & -\sin j\phi \end{pmatrix}$$

$$C_j = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix} \quad \text{and} \quad D_j = \begin{pmatrix} 0 & \xi^{-j} \\ \xi^j & 0 \end{pmatrix},$$

for $j = 0, 1, \ldots, n-1$, where $\phi = 2\pi/n$ and $\xi = e^{i\phi}$ is a primitive root of unity. **(5)**

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Problem 3. Count the number of subgroups of order 6 in S_5 that are isomorphic to S_3 and the number of cyclic subgroups of order 6. **(5)**

Problem 4. Determine the order of the subgroup of the symmetric group S_5 generated by the two permutations (123) and $(24)(35)$. **(5)**

Problem 5. Show that an associative binary structure on a set S which has a left unit and a left inverse of any element a is in fact a group, i.e., has a two-sided unit and a two-sided inverse of any element. **(5)**

Problem 6. Show that the set H of upper triangular matrices form a subgroup in the general linear group $\mathrm{Gl}_n(\mathbb{R})$ of invertible real $n \times n$-matrices. Furthermore, show that the only elements of H that has finite order have order 2. If possible, find all such elements in the cases $n = 2$ and $n = 3$. **(5)**

Problem 7. Let $G \subseteq \mathrm{Gl}_2(\mathbb{F}_3)$ be the subgroup of upper triangular matrices in general linear group over the field $\mathbb{F}_3$ with three elements. [1] Choose a suitable generator set for G and use it to draw the Cayley digraph for G. **(5)**

[1] $\mathbb{F}_3$ can be thought of as $\mathbb{Z}_3$, i.e., the integers modulo 3.